Chapter 8 • Derivatives of Exponential and Logarithmic Functions

Review of Prerequisite Skills

4. **a.** $\log_2 32$

Since $32 = 2^5$, $\log_2 32 = 5$.

b. $\log_{10} 0.0001$

Since $0.0001 = 10^{-4}$, $\log_{10} 0.0001 = -4$.

c. $\log_{10} 20 + \log_{10} 5$

$= \log_{10}(20 \times 5)$

$= \log_{10} 100$

$= \log_{10} 10^2$

$= 2$

d. $\log_2 20 - \log_2 5$

$= \log_2\left(\frac{20}{5}\right)$

$= \log_2 4$

$= 2$

e. $3^{2\log_3 5}$

$= (3^{\log_3 5})^2$

$= 5^2$

$= 25$

f. $\log_3(5^3 9^{-3} 25^{-\frac{3}{2}})$

$= \log_3 5^3 + \log_3 9^{-3} + \log_3 25^{-\frac{3}{2}}$

$= \log_3 5^3 + \log_3 3^{-6} + \log_3 5^{-3}$

$= 3\log_3 5 - 6 - 3\log_3 5$

$= -6$

5. **a.** $\log_2 80$, $b = e$

$= \frac{\log_e 80}{\log_e 2}$

$= \frac{\ln 80}{\ln 2}$

$\doteq 6.322$

b. $3\log_5 22 - 2\log_5 15$, $b = 10$

$= 3\left(\frac{\log_{10} 22}{\log_{10} 5}\right) - 2\left(\frac{\log_{10} 15}{\log_{10} 5}\right)$

$= \frac{3\log_{10} 22 - 2\log_{10} 15}{\log_{10} 5}$

$\doteq 2.397$

Exercise 8.1

4. **c.** $f(x)\ \frac{e^{-x^3}}{x}$

$f'(x) = \frac{-3x^2 e^{-x^3}(x) - e^{-x^3}}{x^2}$

d. $s = \frac{e^{3t^2}}{t^2}$

$\frac{ds}{dt} = 6te^{3t^2}(t^2) - 2t(e^{3t^2})$

$= \frac{2e^{3t^2}[3t^2 - 1]}{t^3}$

h. $g(t) = \frac{e^{2t}}{1 + e^{2t}}$

$g'(t) = \frac{2e^{2t}(1 + e^{2t}) - 2e^{2t}(e^{2t})}{(1 + e^{2t})^2}$

$= \frac{2e^{2t}}{(1 + e^{2t})^2}$

5. **a.** $f'(x) = \frac{1}{3}(3e^{3x} - 3e^{-3x})$

$= e^{3x} - e^{-3x}$

$f'(1) = e^3 - e^{-3}$

c. $h'(z) = 2z(1 + e^{-z}) + z^2(-e^{-z})$

$h'(-1) = 2(-1)(1 + e) + (-1)^2(-e^1)$

$= -2 - 2e - e$

$= -2 - 3e$

7. $y = e^x$

Slope of the tangent is $\frac{dy}{dx} = e^x$.

Slope of the given line is -3.

Slope of the perpendicular line is $\frac{1}{3}$.

Therefore, $e^x = \frac{1}{3}$:

$x \ln e = \ln 1 - \ln 3$

$x = -\ln 3$

$\doteq -1.099$.

The point where the tangent meets the curve has $x = -\ln 3$ and $y = 3^{-\ln 3}$

$= \frac{1}{3}$.

The equation of the tangent is

$y - \frac{1}{3} = \frac{1}{3}(x + \ln 3)$ or $y = 0.3x + 0.6995$.

8. The slope of the tangent line at any point is given by

$\frac{dy}{dx} = (1)(e^{-x}) + x(-e^{-x})$

$= e^{-x}(1 - x)$.

At the point $(1, e^{-1})$, the slope is $e^{-1}(0) = 0$. The equation of the tangent line at the point A is

$y - e^{-1} = 0(x - 1)$ or $y = \frac{1}{e}$.

9. The slope of the tangent line at any point on the curve is $\frac{dy}{dx} = 2xe^{-x} + x^2(-e^{-x})$

$= (2x - x^2)(e^{-x})$

$= \frac{2x - x^2}{e^x}$.

Horizontal lines have slope equal to 0.

We solve $\frac{dy}{dx} = 0$

$\frac{x(2 - x)}{e^x} = 0$.

Since $e^x > 0$ for all x, the solutions are $x = 0$ and $x = 2$. The points on the curve at which the tangents are horizontal are $(0, 0)$ and $\left(2, \frac{4}{e^2}\right)$.

10. If $y = \frac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$,

then $y' = \frac{5}{2}\left(\frac{1}{5}e^{\frac{x}{5}} - \frac{1}{5}e^{-\frac{x}{5}}\right)$,

and $y'' = \frac{5}{2}\left(\frac{1}{25}e^{\frac{x}{5}} + \frac{1}{25}e^{-\frac{x}{5}}\right)$

$= \frac{1}{25}\left[\frac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})\right]$

$= \frac{1}{25}y$.

11. b. $\frac{d^ny}{dx^n} = (-1)^n(3^n)e^{-3x}$

12. In this question, y is an implicitly defined function of x.

a. $\frac{dy}{dx} - \frac{de^{xy}}{dx} = 0$

$\frac{dy}{dx} - e^{xy}\left((1)y + x\frac{dy}{dx}\right) = 0$

$\frac{dy}{dx} - ye^{xy} - xe^{xy}\frac{dy}{dx} = 0$

At the point $(0, 1)$, we get

$\frac{dy}{dx} - 1 - 0 = 0$ and $\frac{dy}{dx} = 1$.

The equation of the tangent line at $A(0, 1)$ is $y - 1 = x$ or $y = x + 1$.

b. $\frac{d}{dx}(x^2e^y) = 0$

$2xe^y + x^2e^y\frac{dy}{dx} = 0$

At the point $(1, 0)$, we get

$2 + \frac{dy}{dx} = 0$

and $\frac{dy}{dx} = -2$.

The equation of the tangent line at $B(1, 0)$ *is* $y = -2(x - 1)$ or $2x + y - 2 = 0$.

c. It is difficult to determine y as an explicit function of x.

13. a. When $t = 0$, $N = 1000[30 + e^0] = 31\,000$.

b. $\frac{dN}{dt} = 1000\left[0 - \frac{1}{30}e^{\frac{-t}{30}}\right] = -\frac{100}{3}e^{-\frac{t}{30}}$

c. When $t = 20h$, $\frac{dN}{dt} = -\frac{100}{3}e^{-\frac{2}{3}} \doteq -17$ bacteria/h.

d. Since $e^{-\frac{t}{30}} > 0$ for all t, *t*here is no solution to $\frac{dN}{dt} = 0$.

Hence, the maximum number of bacteria in the culture occurs at an endpoint of the interval of domain.

When $t = 50$, $N = 1000[30 + e^{-\frac{5}{3}}] \doteq 30\ 189$.
The largest number of bacteria in the culture is 31 000 at time $t = 0$.

14. a. $v = \frac{ds}{dt} = 160\left(\frac{1}{4} - \frac{1}{4}e^{-\frac{t}{4}}\right)$

$= 40(1 - e^{\frac{t}{4}})$

b. $a = \frac{dv}{dt} = 40\left(\frac{1}{4}e^{-\frac{t}{4}}\right) = 10e^{-\frac{t}{4}}$

From **a.**, $v = 40(1 - e^{-\frac{t}{4}})$,

which gives $e^{-\frac{t}{4}} = 1 - \frac{v}{40}$.

Thus, $a = 10\left(1 - \frac{v}{40}\right) = 10 - \frac{1}{4}v$.

c. $v_T = \lim_{t\to\infty} v$

$v_T = \lim_{t\to\infty} 40(1 - e^{-\frac{t}{4}})$

$= 40 \lim_{t\to\infty}\left(1 - \frac{1}{e^{\frac{t}{4}}}\right)$

$= 40(1)$, since $\lim_{t\to\infty} \frac{1}{e^{\frac{t}{4}}} = 0$

The terminal velocity of the skydiver is 40 m/s.

d. Ninety-five per cent of the terminal velocity is

$\frac{95}{100}(40) = 38$ m/s.

To determine when this velocity occurs, we solve

$40(1 - e^{-\frac{t}{4}}) = 38$

$1 - e^{-\frac{t}{4}} = \frac{38}{40}$

$e^{-\frac{t}{4}} = \frac{1}{20}$

$e^{\frac{t}{4}} = 20$

and $\frac{t}{4} = \ln 20$,

which gives $t = 4$

$\ln 20 \doteq 12$ s.

The skydiver's velocity is 38 m/s, 12 s after jumping.
The distance she has fallen at this time is

$S = 160(\ln 20 - 1 + e^{-\ln 20})$

$= 160\left(\ln 20 - 1 + \frac{1}{20}\right)$

$\doteq 327.3$ m.

15. a. The given limit can be rewritten as

$\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} \frac{e^{0+h} - e^0}{h}$

This expression is the limit definition of the derivative at $x = 0$ for $f(x) = e^x$.

$\left[f'(0) = \lim_{h\to 0} \frac{e^{0+h} - e^0}{h}\right]$

Since $f'(x) = \frac{de^x}{dx} = e^x$, the value of the given limit is $e^0 = 1$.

b. Again, $\lim_{h\to 0} \frac{e^{2+h} - e^2}{h}$ is the derivative of e^x at $x = 2$.

Thus, $\lim_{h\to 0} \frac{e^{2+h} - e^2}{h} = e^2$.

16. For $y = Ae^{mt}$, $\frac{dy}{dt} = Ame^{mt}$ and $\frac{d^2y}{dt^2} = Am^2e^{mt}$.

Substituting in the differential equation gives

$Am^2e^{mt} + Ame^{mt} - 6Ae^{mt} = 0$

$Ae^{mt}(m^2 + m - 6) = 0$.

Since $Ae^{mt} \neq 0$, $m^2 + m - 6 = 0$

$(m + 3)(m - 2) = 0$

$m = -3$ or $m = 2$.

17. a. $D_x \sinh x = \cosh x$

$D_x \sinh x = D_x\left[\frac{1}{2}(e^x - e^{-x})\right]$

$= \frac{1}{2}(e^x + e^{-x}) = \cosh x$

b. $D_x \cosh x = \sinh x$

$D_x \cosh x = \frac{1}{2}(e^x - e^{-x}) = \sinh x$

c. $D_x \tanh x = \frac{1}{(\cosh x)^{2g}}$

$\tanh x = \frac{\sinh x}{\cosh x}$

Since $\tanh x = \frac{\sinh x}{\cosh x}$,

$D_x \tanh x = \frac{(D_x \sinh x)(\cosh x) - (\sinh x)(D_x \cosh x)}{(\cosh x)^2}$

$= \frac{\frac{1}{2}\left(e^x + e^{-x}\right)\left(\frac{1}{2}\right)\left(e^x + e^{-x}\right) - \frac{1}{2}\left(e^x - e^{-x}\right)\left(\frac{1}{2}\right)\left(e^x - e^{-x}\right)}{(\cosh x)^2}$

$= \frac{\frac{1}{4}\left[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right]}{(\cosh x)^2}$

$= \frac{\frac{1}{4}(4)}{(\cosh x)^2}$

$= \frac{1}{(\cosh x)^2}$

Exercise 8.2

2. Since $e = \lim\limits_{h\to 0}(1+h)^{\frac{1}{h}}$, let $h = \frac{1}{n}$. Therefore,

$$e = \lim_{\frac{1}{n}\to 0}\left(1+\frac{1}{n}\right)^{n}.$$

But as $\frac{1}{n}\to 0$, $n\to\infty$.

Therefore, $e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$.

$$\text{If } n = 100,\ e \doteq \left(1+\frac{1}{100}\right)^{100}$$
$$= 1.01^{100}$$
$$\doteq 2.70481.$$

Try $n = 100\,000$, etc.

4. f. $g(z) = \ln(e^{-z}+ze^{-z})$

$$g'(z) = \frac{1}{e^{-z}+ze^{-z}}\left[-e^{-z}+(e^{-z}-ze^{-z})\right]$$
$$= \frac{-ze^{-z}}{e^{-z}+ze^{-z}}$$

h. $h(u) = e^{\sqrt{u}}\ln u^{\frac{1}{2}}$

$$= e^{\sqrt{u}}\left(\frac{1}{2}\ln u\right)$$
$$h'(u) = e^{\sqrt{u}}\left(\frac{1}{2\sqrt{u}}\right)\left(\frac{1}{2}\ln u\right)+\frac{1}{2}\left(\frac{1}{u}\right)e^{\sqrt{u}}$$
$$= \frac{1}{2}e^{\sqrt{u}}\left(\frac{1}{2}e^{1\sqrt{u}}\ln u+\frac{1}{u}\right)$$

i. $f(x) = \ln\left(\frac{x^2+1}{x-1}\right)$

$$f'(x) = \frac{1}{\frac{x^2+1}{x-1}}\left[\frac{2x(x-1)-(x^2+1)}{(x-1)^2}\right]$$
$$= \frac{x-1}{x^2+1}\left[\frac{x^2-2x-1}{(x-1)^2}\right]$$
$$= \frac{x^2-2x-1}{(x-1)(x^2+1)}$$

5. a. $g(x) = e^{2x-1}\ln(2x-1)$

$$g'(x) = e^{2x-1}(2)\ln(2x-1)+\left(\frac{1}{2x-1}\right)(2)e^{2x-1}$$
$$g'(1) = e^{2}(2)\ln(1)+1(2)e^{1}$$
$$= 2e$$

b. $f(t) = \ln\left(\frac{t-1}{3t+5}\right)$

$$f'(t) = \left(\frac{3t+t}{t-1}\right)\left[\frac{3t+5-3(t-1)}{(3t+5)^2}\right]$$
$$f'(5) = \frac{20}{4}\left[\frac{20-12}{20^2}\right]$$
$$= \frac{8}{4\times 20}$$
$$= \frac{1}{10}$$
$$= 0.1$$

6. a. $f(x) = \ln(x^2+1)$

$$f'(x) = \left(\frac{1}{1+x^2}\right)(2x)$$
$$= \frac{2x}{1+x^2}$$

Since $1+x^2>0$ for all x, $f'(x)=0$ when $2x=0$, i.e., when $x=0$.

b. $f(x) = (\ln x+2x)^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3}(\ln x+2x)^{-\frac{2}{3}}\left(\frac{1}{x}+2\right)$$
$$= \frac{\frac{1}{x}+2}{3(\ln x+2x)^{\frac{2}{3}}}$$

$f'(x)=0$ if $\frac{1}{x}+2=0$ and $(\ln x+2x)^{\frac{2}{3}}\neq 0$.

$\frac{1}{x}+2=0$ when $x=-\frac{1}{2}$.

Since $f(x)$ is defined only for $x>0$, there is no solution to $f'(x)=0$.

c. $f(x) = (x^2+1)^{-1}\ln(x^2+1)$

$$f'(x) = -(x^2+1)^{-2}(2x)\ln(x^2+1)+(x^2+1)^{-1}\left(\frac{2x}{x^2+1}\right)$$
$$= \frac{2x(1-\ln(x^2+1))}{(x^2+1)^2}$$

Since $(x^2+1)^2\geq 1$ for all x, $f'(x)=0$, when $2x(1-\ln(x^2+1))=0$.

Hence, the solution is

$x=0$ or $\ln(x^2+1)=1$

$x^2+1=e$

$x=\pm\sqrt{e-1}$.

7. a. $f(x) = \dfrac{\ln \sqrt[3]{x}}{x}$

$= \dfrac{\frac{1}{3}\ln x}{x}$

$f'(x) = \dfrac{1}{3}\left[\dfrac{\frac{1}{x} \bullet x - \ln x}{x^2}\right]$

At the point (1, 0), the slope of the tangent line is

$f'(1) = \dfrac{1}{3}\left[\dfrac{1-0}{1}\right]$

$= \dfrac{1}{3}.$

The equation of the tangent line is $y = \dfrac{1}{3}\left(x - 1\right)$

or $x - 3y - 1 = 0$.

b. Use the $\boxed{y =}$ button to define $f(x)$ and set the window so $-1 \le x \le 4$ and $-2 \le y \le 0.5$.
Select $\boxed{2^{\text{ND}}}$ $\boxed{\text{DRAW}}$ and pick menu item five to draw the tangent at the point (1, 0).

c. The calculator answer is $y = 0.31286x - 0.31286$.
This can be improved using the $\boxed{\text{ZOOM}}$ feature.

8. The line defined by $3x - 6y - 1 = 0$ has slope $\dfrac{1}{2}$.

For $y = \ln x - 1$, the slope at any point is $\dfrac{dy}{dx} = \dfrac{1}{2}$.

Therefore, at the point of tangency $\dfrac{1}{x} = \dfrac{1}{2}$,

or $x = 2$ and $y = \ln 2 - 1$.

The equation of the tangent is

$y - (\ln 2 - 1) = \dfrac{1}{2}\left(x - 2\right)$

or $x - 2y + (2\ln 2 - 4) = 0$

9. a. For a horizontal tangent line, the slope equals 0.
We solve:

$f'(x) = 2(x \ln x)\left(\ln x + x \bullet \dfrac{1}{x}\right) = 0$

$x = 0$ or $\ln x = 0$ or $\ln x = -1$

No ln in the domain $x = 1$ $x = e^{-1} = \dfrac{1}{e}$

The points on the graph of $f(x)$ at which there are horizontal tangents are $\left(\dfrac{1}{e}, \dfrac{1}{e^2}\right)$ and (1, 0).

b. Graph the function and use the $\boxed{\text{TRACE}}$ and $\boxed{\text{CALC}}$ $\dfrac{dy}{dx}$ features to determine the points where $\dfrac{dy}{dx} = 0$.

c. The solution in **a.** is more precise and efficient.

11. $v(t) = 90 - 30\ln(3t + 1)$

a. At $t = 0$, $v(0) = 90 - 30\ln(1) = 90$ km/h.

b. $a = v'(t) = \dfrac{-30}{3t+1} \bullet 3 = \dfrac{-90}{3t+1}$

c. At $t = 2$, $a = -\dfrac{90}{7} \doteq -12.8$ km/h/s.

d. The car is at rest when $v = 0$.
We solve:

$v(t) = 90 - 30\ln(3t + 1) = 0$

$\ln(3t + 1) = 3$

$3t + 1 = e^3$

$t = \dfrac{e^3 - 1}{3} = 6.36$ s.

12. a. $\text{pH} = -\log_{10}(6.3 \times 10^{-5})$

$= -[\log_{10} 6.3 + \log_{10} 10^{-5}]$

$\doteq -[0.7993405 - 5]$

$\doteq 4.20066$

The pH value for tomatoes is approximately 4.20066.

b. $H(t) = 30 - 5t - 25(e^{-\frac{t}{5}} - 1)$

$\text{pH} = -\dfrac{\ln(30 - 5t - 25(e^{-\frac{t}{5}} - 1))}{\ln 10}$

$= -\dfrac{1}{\ln 10}\ln(55 - 5t - 25e^{-\frac{t}{5}})$

$\dfrac{d}{dt}\text{pH} = -\dfrac{1}{\ln 10} \bullet \dfrac{-5 + 5e^{-\frac{t}{5}}}{55 - 5t - 25e^{-\frac{t}{5}}}$

$= -\dfrac{1}{\ln 10} \bullet \dfrac{-1 + e^{-\frac{t}{5}}}{11 - t - 5e^{-\frac{t}{5}}}$

When $t = 10$ s, $\dfrac{d}{dt}\text{pH} = -\dfrac{1}{\ln 10} \bullet \dfrac{-1 + e^{-2}}{1 - 5e^{-2}}$

$= \dfrac{1}{\ln 10} \bullet \dfrac{e^2 - 1}{e^2 - 5}$

$\doteq 1.16.$

13. $\dfrac{d^2F}{dS^2} = F - 18ke^{-2S}$

$F = k(e^{-s} - 6e^{-2S})$

$\dfrac{dF}{dS} = k(-e^{-s} + 12e^{-2S})$

$\dfrac{d^2F}{dS^2} = k(e^{-s} - 24e^{-2S})$

$= k(e^{-s} - 6e^{-2S} - 18e^{-2S})$

$= k(e^{-S} - 6e^{-2S}) - 18ke^{-2S}$

$= F - 18ke^{-2S}$

14. a. We assume y is an implicitly defined function of x, and differentiate implicitly with respect to x.

$$(1)(e^4) + x(e^y)\frac{dy}{dx} + \frac{dy}{dx}\ln x + y\left(\frac{1}{x}\right) = 0$$

At the point (1, ln 2) the derivative equation simplifies to

$$(1)(e^{\ln 2}) + (1)(e^{\ln 2})\frac{dy}{dx} + \frac{dy}{dx}\ln(1) + \ln 2(1) = 0$$

$$2 + 2\frac{dy}{dx} + 0 + \ln 2 = 0$$

$$\frac{dy}{dx} = \frac{-2 - \ln 2}{2}$$

The slope of the tangent to the curve at (1, ln 2) is $-\dfrac{2 + \ln 2}{2}$

b. $\ln\sqrt{xy} = 0$

$$\frac{1}{2}\ln(xy) = 0$$

$$\ln(xy) = 0$$

$$xy = e^0 = 1$$

$$y = \frac{1}{x}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

The slope of the tangent to the curve at $\left(\frac{1}{3}, 3\right)$ is –9.

15. By definition, $\dfrac{d}{dx}\ln x = \lim\limits_{h \to 0}\dfrac{\ln(2 + h) - \ln 2}{h}$

$$= \frac{1}{x}.$$

The derivative of ln x at $x = 2$ is

$$\lim_{h \to 0}\frac{\ln(2 + h) - \ln 2}{h} = \frac{1}{2}.$$

16. a.

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \dots$$

$$= 1 + 1 + \frac{1}{2!}(1)\left(1 - \frac{1}{n}\right) + \frac{1}{3!}(1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots$$

$$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!}\lim_{n \to \infty}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\lim_{n \to \infty}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots$$

Thus, $e = 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dots$

b. $S_3 = 1 + 1 + \dfrac{1}{2!} = 2.5$

$$S_4 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} = 2.\dot{6}$$

$$S_5 = S_4 + \frac{1}{4!} = 2\,\frac{2}{3} + \frac{1}{24} = 2.708\dot{3}$$

$$S_7 = S_6 + \frac{1}{6!} = 2.71\dot{6} + \frac{1}{720} = 2.7180\dot{5}$$

17. a. $y = \ln|x| = \begin{cases}\ln x, \text{ if } x > 0\\ \ln(-x), \text{ if } x < 0\end{cases}$

$$\frac{dy}{dx} = \begin{cases}\dfrac{1}{x}, \text{ if } x > 0\\ \dfrac{1}{-x} \bullet -1, \text{ if } x < 0\end{cases}$$

$$= \begin{cases}\dfrac{1}{x}, \text{ if } x > 0\\ \dfrac{1}{x}, \text{ if } x < 0\end{cases}$$

Thus, $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$ for all $x \neq 0$.

b. $\dfrac{dy}{dx} = \dfrac{1}{2x + 1} \bullet 2 = \dfrac{2}{2x + 1}$

c. $\dfrac{dy}{dx} = 2x\ln|x| + x^2\left(\dfrac{1}{x}\right)$

$$= 2x\ln|x| + x$$

Exercise 8.3

2. e. $f(x) = \dfrac{3^{\frac{x}{2}}}{x^2}$

$$f'(x) = \frac{\frac{1}{2}\ln 3(3^{\frac{x}{2}})(x^2) - 2x(3^{\frac{x}{2}})}{x^4}$$

$$= \frac{x\ln 3(3^{\frac{x}{2}}) - 4(3^{\frac{x}{2}})}{2x^4}$$

$$= \frac{3^{\frac{x}{2}}[x\ln 3 - 4]}{2x^4}$$

f. $\dfrac{\log_5(3x^2)}{\sqrt{x + 1}}$

$$f'(x) = \frac{\frac{1}{\ln 5(3x^2)}(6x)(\sqrt{x + 1}) - \frac{1}{2}(x + 1)^{-\frac{1}{2}}(\log_5 3x^2)}{x + 1}$$

3. a. $f(t) = \log_2\left(\dfrac{t+1}{2t+7}\right)$

$f(t) = \log_2(t+1) - \log_2(2t+1)$

$f'(t) = \dfrac{1}{(t+1)\ln 2} - \dfrac{2}{(2t+1)\ln 2}$

$f'(3) = \dfrac{1}{4\ln 2} - \dfrac{2}{13\ln 2}$

$= \dfrac{13}{52\ln 2} - \dfrac{8}{52\ln 2}$

$= \dfrac{5}{52\ln 2}$

b. $h(t) = \log_3[\log_2(t)]$

$h'(t) = \dfrac{1}{\ln 3(\log_2 t)} \times \dfrac{1}{\ln 2(t)}$

$h'(8) = \dfrac{1}{\ln 3\log_2 8} \times \dfrac{1}{8(\ln 2)}$

$= \dfrac{1}{3\ln 3} \times \dfrac{1}{8\ln 2}$

$= \dfrac{1}{24(\ln 3)(\ln 2)}$

5. a. $\dfrac{dy}{dx} = \log_{10}(x^2 - 3x)(\ln 10(10^{2x-9})2) + 10^{2x-9}\left[\dfrac{1(2x-3)}{\ln 10\,(x^2-3x)}\right]$

At $x = 5$, $\dfrac{dy}{dx} = 2\log_{10}10[\ln 10(10)) + 10\left[\dfrac{7}{\ln 10}\right]$

$= 20\ln 10 + \dfrac{7}{\ln 10}$

$\doteq 49.1.$

When $x = 5$, $y = 10(\log_{10}10)$

$= 10.$

Equation is $y - 10 = 49.1(x - 5)$ or $y = 49.1x - 237.5$.

b. When using a graphing calculator it is necessary to use the $\boxed{\text{ZOOM}}$ function to get the x-coordinate close to 5.

6. $y = 20 \times 10\left(\dfrac{t-5}{10}\right)$

To find the point where the curve crosses the y-axis, set $t = 0$.

Thus, $y = 20(10^{-\frac{1}{2}})$

$= \dfrac{20}{\sqrt{10}}$

$= 2\sqrt{10}.$

The point of tangency is $(0, 2\sqrt{10})$.

The slope of the tangent is given by

$\dfrac{dy}{dx} = 20\left(10^{\frac{t-5}{10}}\right)(\ln 10)\left(\dfrac{1}{10}\right).$

At $(0, 2\sqrt{10})$ the slope of the tangent is $\dfrac{2\ln 10}{\sqrt{10}}$.

The equation of the tangent is $y - 2\sqrt{10} = \dfrac{2\ln 10}{10}(x - 0)$

or $2\ln 10x - \sqrt{10}y + 20 = 0$

7. a. For $f(x) = \log_2(\log_2(x))$ to be defined, $\log_2 x > 0$.

For $\log_2 x > 0$, $x > 2^0 = 1$.

Thus, the domain of $f(x)$ *is* $x > 1$.

b. The x-intercept occurs when $f(x) = 0$.

Thus, $\log_2(\log_2 x) = 0$

$\log_2 x = 1$

$x = 2.$

The slope of the tangent is given by

$f'(x) = \dfrac{1}{(\log_2 x)(\ln 2)} \bullet \dfrac{1}{x\ln 2}.$

At $x = 2$, the slope is

$f'(2) = \dfrac{1}{(1)(\ln 2)} \bullet \dfrac{1}{2\ln 2}$

$= \dfrac{1}{2(\ln 2)^2}.$

c. It is difficult to directly graph logarithm functions having a base other than 10 or e.

8. $s = 40 + 3t + 0.01t^2 + \ln t$

a. $$v = \frac{ds}{dt} = 3 + 0.02t + \frac{1}{t}$$

When $t = 20$, $v = 3 + 0.4 + 0.05$
$= 3.45$ cm/min.

b. $$a = \frac{dv}{dt} = 0.02 - \frac{1}{t^2}$$

When $a = 0.01$,

$$0.02 - \frac{1}{t^2} = 0.01$$

$$\frac{1}{t^2} = 0.01$$

$$t^2 = 100$$

$$t = 10.$$

After 10 minutes, the acceleration is 10 cm/min/min.

9. $P = 0.5(10^9)\,e^{0.20015t}$

a. $$\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{0.20015t}$$

In 1968, $t = 1$ and $\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{0.20015}$

$\doteq 0.12225 \times 10^9$ dollars/annum.

In 1978, $t = 11$ and $\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{11 \times 0.20015}$

$\doteq 0.90467 \times 10^9$ dollars/annum.

In 1978, the rate of increase of debt payments was \$904,670,000/annum compared to \$122,250,000/annum in 1968.

b. In 1988, $t = 21$ and $\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{21 \times 0.20015}$

$\doteq 6.69469 \times 10^9$ dollars/annum.

In 1998, $t = 31$ and $\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{31 \times 0.20015}$

$\doteq 49.5417 \times 10^9$ dollars/annum.

Note the continuing increase in the rates of increase of the debt payments.

10. a. For an earthquake of intensity I,

$$R = \log_{10}\left(\frac{I}{I_0}\right).$$

For an earthquake of intensity $10I$,

$$R = \log_{10}\left(\frac{10I}{I_0}\right).$$

$$= \log_{10}\left(\frac{I}{I_0}\right) + \log_{10}10$$

$$= \log_{10}\left(\frac{I}{I_0}\right) + 1.$$

The Richter magnitude of an earthquake of intensity $10I$ *is* 1 greater than that of intensity I.

b. $R = \log_{10}I - \log_{10}I_0$

$$\frac{dR}{dt} = \frac{1}{I \ln 10} \bullet \frac{dI}{dt} - 0$$

We are given that $\frac{dI}{dt} = 100$ and $I = 35$.

Thus, $\frac{dR}{dt} = \frac{1}{35 \ln 10} \bullet 100 \doteq 1.241$ units/s.

When the intensity of an earthquake is 35 and increasing at the rate of 100 units/s, the Richter magnitude is increasing at the rate of 1.24 units/s.

11. b. Rewrite $y = 7^x$ *as* $y = e^{x\ln 7}$ and graph using $\boxed{y =}$.

c. The factor ln 7 is a power used to transform $y = e^x$ to $y = (e^x)^{\ln 7}$.

12. b. Rewrite $y = \log_5 x$ *as* $y = \frac{\ln x}{\ln 5} = \frac{1}{\ln 5} \ln x$, and graph using $\boxed{y =}$.

c. Since $\frac{1}{\ln 5} < 1$, multiplying $\ln x$ by $\frac{1}{\ln 5}$ causes the graph of $y = \ln x$ to be compressed vertically.

Exercise 8.4

1. a. $f(x) = e^{-x} - e^{-3x}$ on $0 \leq \times \leq 10$

$f'(x) = -e^{-x} + 3e^{-3x}$

Let $f'(x) = 0$, therefore $e^{-x} + 3e^{-3x} = 0$.

Let $e^{-x} = w$, when $-w + 3w^3 = 0$.

$w(-1 + 3w^2) = 0$.

Therefore, $w = 0$ or $w^2 = \frac{1}{3}$

$w = \pm \frac{1}{\sqrt{3}}$.

But $w \geq 0$, $w = +\frac{1}{\sqrt{3}}$.

When $w = \frac{1}{\sqrt{3}}$, $e^{-x} = \frac{1}{\sqrt{3}}$,

$-x \ln e = \ln 1 - \ln\sqrt{3}$

$x = \frac{\ln\sqrt{3} - \ln 1}{1}$

$= \ln\sqrt{3}$

$\doteq 0.55$.

$f(0) = e^0 - e^0$

$= 0$

$f(0.55) \doteq -4.61$

$f(100) = e^{-100} - e^{-300} \doteq 3.7$

Absolute maximum is about 3.7 and absolute minimum is about –4.61.

b. $g(t) = \frac{e^t}{1 + \ln t}$ on $1 \leq t \leq 12$

$$g'(t) = \frac{e^t(1 + \ln t) - \frac{1}{t}(e^t)}{(1 + \ln t)^2}$$

Let $g'(t) = 0$:

$e^t(1 + \ln t) - \frac{1}{t}(e^t) = 0$

Since $e^t \neq 0$, $1 + \ln t - \frac{1}{t} = 0$,

$\ln t = \frac{1}{t} - 1$.

Therefore, $t = 1$ by inspection.

$g(1) = \frac{e^1}{1 + \ln 1} = 2.7$

$g(12) = \frac{e^{12}}{1 + \ln 12} \doteq 46\ 702$

The maximum value is about 46 202 and the minimum value is 2.7.

c. $m(x) = (x + 2)e^{-2x}$ on $-4 \leq x \leq 4$

$m'(x) = e^{-2x} + (-2)(x + 2)e^{-2x}$

Let $m'(x) = 0$.

$e^{-2x} \neq 0$, therefore, $1 + (-2)(x + 2) = 0$

$x = \frac{-3}{2}$

$= -1.5$.

$m(-4) = -2e^8 \doteq -5961$

$m(-1.5) = 0.5e^3 \doteq 10$

$m(4) = 6e^{-8} \doteq 0.0002$

The maximum value is about 10 and the minimum value is about –5961.

d. $s(t) = \ln\left(\frac{t^2 + 1}{t^2 - 1}\right) + 6 \ln t$ on $1.1 \leq t \leq 10$

$= \ln(t^2 + 1) - \ln(t^2 - 1) + 6 \ln t$

$s'(t) = \frac{2t}{t^2 + 1} - \frac{2t}{t^2 - 1} + \frac{6}{t}$

Let $s'(t) = 0$,

$\frac{-4t + 6(t^2 - 1)(t^2 + 1)}{t(t^2 + 1)(t^2 - 1)} = 0$ or $-4t + 6(t^4 - 1) = 0$

$3t^4 - 2t - 3 = 0$

$t \doteq 1.2$ (using a calculator).

$s(1.1) = \ln\left(\frac{1.1^2 + 1}{1.1^2 - 1}\right) + 6 \ln(1.1) \doteq 2.9$

$s(1.2) = \ln\left(\frac{1.2^2 + 1}{1.2^2 - 1}\right) + 6 \ln(1.2) \doteq 2.8$

$s(10) = \ln\left(\frac{10^2 + 1}{10^2 - 1}\right) + 6 \ln(10) \doteq 13.84$

The maximum value is about 13.84 and the minimum is about 2.8.

4. a. $P(x) = 10^6[1 + (x - 1)e^{-0.001x}]$, $0 \leq x \leq 2000$

Using the Algorithm for Extreme Values, we have

$P(0) = 10^6[1 - 1] = 0$

$P(2000) = 10^6[1 + 1999e^{-2}] \doteq 271.5 \times 10^6$.

Now,

$P'(x) = 10^6\left[(1)e^{-0.001x} + (x - 1)(-0.001)e^{-0.001x}\right]$

$= 10^6e^{-0.001x}(1 - 0.001x + 0.001)$

Since $e^{-0.001x} > -$ for all x,

$P'(x) = 0$ when $1.001 - 0.001x = 0$

$x = \dfrac{1.001}{0.001} = 1001.$

$P(1001) = 10^6[1 + 1000e^{-1.001}] \doteq 368.5 \times 10^6$

The maximum monthly profit will be 368.5×106 dollars when 1001 items are produced and sold.

b. The domain for $P(x)$ becomes $0 \leq x \leq 500$.

$P(500) = 10^6[1 + 499e^{-0.5}] = 303.7 \times 10^6$

Since there are no critical values in the domain, the maximum occurs at an endpoint. The maximum monthly profit when 500 items are produced and sold is 303.7×10^6 dollars.

5. $R(x) = 40x^2e^{-0.4x} + 30$, $0 \leq x \leq 8$

We use the Algorithm for Extreme Values:

$R'(x) = 80xe^{-0.4x} + 40x^2(-0.4)e^{-0.4x}$

$= 40xe^{-0.4x}(2 - 0.4x)$

Since $e^{-0.4x} > 0$ for all x, $R'(x) = 0$ when

$x = 0$ or $2 - 0.4x = 0$

$x = 5.$

$R(0) = 30$

$R(5) \doteq 165.3$

$R(8) \doteq 134.4$

The maximum revenue of 165.3 thousand dollars is achieved when 500 units are produced and sold.

6. The speed of the signal is $S(x) = kv(x)$

$= kx^2 \ln\left(\dfrac{1}{x}\right)$

$= kx^2(\ln 1 - \ln x)$

$= -kx^2 \ln x.$

Since $x = \dfrac{r}{R}$, we have $\dfrac{R}{10} \leq r \leq \dfrac{9R}{10}$

$\dfrac{1}{10} \leq \dfrac{r}{R} \leq \dfrac{9}{10}$

$\dfrac{1}{10} \leq x \leq \dfrac{9}{10}.$

Now, $S'(x) = -k\left(2x \ln x + x^2\left(\dfrac{1}{x}\right)\right)$

$= -kx(2 \ln x + 1).$

$S'(x) = 0$ when $\ln x = -\dfrac{1}{2}$ $\left(\text{since } x \geq \dfrac{1}{10}, x \neq 0\right)$

$x = e^{-\frac{1}{2}}$

$\doteq 0.6065.$

$S(0.1) = 0.023k$

$S(e^{-\frac{1}{2}}) = 0.184k$

$S(0.9) = 0.08k$

The maximum speed of the signal is $0.184k$ units when $x \doteq 0.61$.

7. $C(h) = 1 + h(\ln h)^2$, $0.2 \leq h \leq 1$

$C'(h) = (\ln h)^2 + 2h \ln h \bullet \dfrac{1}{h}$

$= (\ln h)^2 + 2 \ln h$

$= \ln h (\ln h + 2)$

$C'(h) = 0$ when $\ln h = 0$ or $\ln h = -2$

$h = 1$ or $h = e^{-2} \doteq 0.135.$

Since the domain under consideration is $0.2 \leq h \leq 0.75$, neither of the critical values is admissible.

$C(0.2) \doteq 1.52$

$C(0.75) \doteq 1.06$

The student's intensity of concentration level is lowest at the 45 minute mark of the study session.

8. $P(t) = 100(e^{-t} - e^{-4t})$, $0 \leq t \leq 3$

$P'(t) = 100(-e^{-t} + 4e^{-4t})$

$= 100e^{-t}(-1 + 4e^{-3t})$

Since $e^{-t} > 0$ for all t, $P'(t) = 0$ when

$4e^{-3t} = 1$

$e^{-3t} = \dfrac{1}{4}$

$-3t = \ln(0.25)$

$t = \dfrac{-\ln(0.25)}{3}$

$= 0.462.$

$P(0) = 0$

$P(0.462) \doteq 47.2$

$P(3) \doteq 4.98$

$P'(t) = 100(4e^{-4t})$

$= \dfrac{400}{e^{4t}} > 0$ for all t

Since there are no critical values in the given interval, the maximum value will occur at an endpoint.
$P(0) = 0$
$P(3) \doteq 4.98$
The highest percentage of people spreading the rumour is 4.98% and occurs at the 3 h point.

9. $C = 0.015 \times 10^9 e^{0.07533t}$

a. $\frac{dp}{dt} = 0.5(10^9)(0.20015)e^{0.20015t}$

In 1968, $t = 1$ and $\frac{dp}{dt} = 0.5(10^9)(0.20015)e^{0.20015}$

$\doteq 0.1225 \times 10^9$ dollars/year

In 1978, $t = 11$ and $\frac{dp}{dt} = 0.5(10^9)(0.20015)e^{11x0.20015}$

$\doteq 0.90467 \times 10^9$ dollars/year

In 1978 the rate of increase of debt payments was \$904 670 000/year compared to \$122 250 000/year in 1968.

b. In 1987, $t = 20$ and $\frac{dp}{dt} = 0.5(10^9)(0.20015)e^{20x0.20015}$

$\doteq 5.48033 \times 10^9$ dollars/year

In 1989, $t = 22$ and $\frac{dp}{dt} = 0.5(10^9)(0.20015)e^{22x0.20015}$

$\doteq 8.17814 \times 10^9$ dollars

c. In 1989, $P = 0.5(10^9)(e^{20x0.20015})$

$\doteq 27.3811 \times 10^9$ dollars

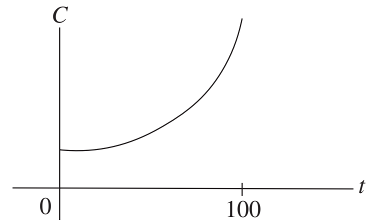
In 1989, $P = 0.5(10^9(e^{22x0.20015})$

$\doteq 40.8601 \times 10^9$ dollars

Year	Amount Paid	Rate of Change	%
1987	\$27.3811 $\times 10^9$	\$5.48033 $\times 10^9$/year	20.02
1989	\$40.8601 $\times 10^9$	\$8.17814/year	20.02

9. $C = 0.015 \times 10^9 e^{0.07533t}$, $0 \leq t \leq 100$

a.



b. $\frac{dC}{dt} = 0.015 \times 10^9 \times 0.07533e^{0.07533t}$

In 1947, $t = 80$ and the growth rate was

$\frac{dC}{dt} = 0.46805 \times 10^9$ dollars/year.

In 1967, $t = 100$ and the growth rate was

$\frac{dC}{dt} = 2.1115 \times 10^9$ dollars/year.

The ratio of growth rates of 1967 to that of 1947 is

$\frac{2.1115 \times 10^9}{0.46805 \times 10^9} = \frac{4.511}{1}$.

The growth rate of capital investment grew from 468 million dollars per year in 1947 to 2.112 billion dollars per year in 1967.

c. In 1967, the growth rate of investment as a percentage of the amount invested is

$\frac{2.1115 \times 10^9}{28.0305 \times 10^9} \times 100 = 7.5\%$.

d. In 1977, $t = 110$

$C = 59.537 \times 10^9$ dollars

$\frac{dC}{dt} = 4.4849 \times 10^9$ dollars/year.

e. Statistics Canada data shows the actual amount of U.S. investment in 1977 was 62.5×10^9 dollars. The error in the model is 3.5%.

f. In 2007, $t = 140$.
The expected investment and growth rates are
$C = 570.490 \times 10^9$ dollars and $\frac{dC}{dt} = 42.975 \times 10^9$ dollars/year.

10. $C(t) = \frac{k}{b-a}(e^{at} - e^{-bt})$, $b > a > 0$, $k > 0$, $t \geq 0$

$C^{1}(t) = \frac{k}{b-a}(-ae^{-at} + be^{-bt})$

$C^{1}(t) = 0$ when $be^{-bt} - ae^{-at} = 0$

$$be^{-bt} = ae^{-at}$$

$$\frac{b}{a} = \frac{e^{bt}}{e^{at}} = e^{(b-a)t}$$

$$(b-a)t = \ln\left(\frac{b}{a}\right)$$

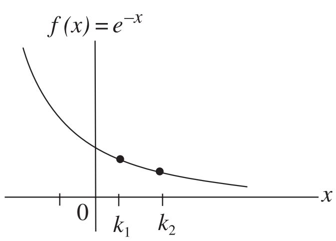
$$t = \frac{\ln(b) - \ln(a)}{b-a}$$

Since $\frac{b}{a} > 1$, $\ln\left(\frac{b}{a}\right) > 0$ and hence the value of t is a positive number. If $t = 0$, $C(0) = \frac{k}{b-a}(1-1) = 0$

Also, $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty}\left[\frac{k}{b-a}\left(\frac{1}{e^{at}} - \frac{1}{e^{bt}}\right)\right]$

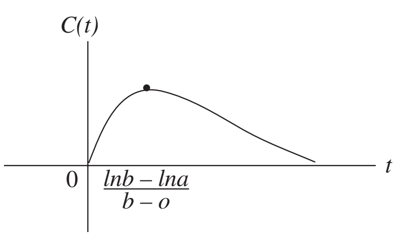
$$= \frac{k}{b-a}(0-0) = 0$$

Since $f(x) = e^{-x}$ is a decreasing function throughout its domain, if $k_1 < k_2$ then $e^{-k_1} > e^{-k_2}$



Since $a < b$, $at < bt$ where a, b, t are all positive.
Thus, $e^{-at} > e^{-bt}$ for all $t > 0$.
Hence, $C(t) > 0$ for all $t > 0$.
Since there is only one critical value, the largest concentration of the drug in the blood occurs at

$t = \frac{\ln b - \ln a}{b-a}$



11. a. The growth function is $N = 2^{\frac{t}{5}}$.
The number killed is given by $K = e^{\frac{t}{3}}$.
After 60 minutes, $N = 2^{12}$.
Let T be the number of minutes after 60 minutes.
The population of the colony at any time, T, after the first 60 minutes is
$P = N - k$

$= 2\,\frac{60+T}{5} - e^{\frac{T}{3}}$, $T \geq 0$

$\frac{dP}{dt} = 2^{\frac{60+T}{5}}\left(\frac{1}{5}\right)\ln 2 - \frac{1}{3}e^{\frac{T}{3}}$

$= 2^{12+\frac{T}{5}}\left(\frac{\ln 2}{5}\right) - \frac{1}{3}e^{\frac{T}{3}}$

$= 2^{12} \bullet 2^{\frac{T}{5}}\left(\frac{\ln 2}{5}\right) - \frac{1}{3}e^{\frac{T}{3}}$

$\frac{dP}{dt} = 0$ when $2^{12}\,\frac{\ln 2}{5}\,2^{\frac{T}{5}} = \frac{1}{3}e^{\frac{T}{3}}$

or $3\,\frac{\ln 2}{5} \bullet 2^{12}\,2^{\frac{T}{5}} = e^{\frac{T}{3}}$.

We take the natural logarithm of both sides:

$\ln\left(3.2^{12}\,\frac{\ln 2}{5}\right) + \frac{T}{5}\ln 2 = \frac{T}{3}$

$$7.4404 = T\left(\frac{1}{3} - \frac{\ln 2}{5}\right)$$

$$T = \frac{7.4404}{0.1947} = 38.2 \text{ min.}$$

At $T = 0$, $P = 2^{12} = 4096$.
At $T = 38.2$, $P = 478\,158$.
For $T > 38.2$, $\frac{dP}{dt}$ is always negative.
The maximum number of bacteria in the colony occurs 38.2 min after the drug was introduced. At this time the population numbers 478 158.

b. $P = 0$ when $2^{\frac{60+T}{5}} = e^{\frac{T}{3}}$

$$\frac{60+T}{5}\ln 2 = \frac{T}{3}$$

$$12\ln 2 = T\left(\frac{1}{3} - \frac{\ln 2}{5}\right)$$

$$T = 42.72$$

The colony will be obliterated 42.72 minutes after the drug was introduced.

12. Let t be the number of minutes assigned to study for the first exam and $30 - t$ minutes assigned to study for the second exam. The measure of study effectiveness for the two exams is given by

$$E(t) = E_1(t) + E_2(30 - t),\ 0 \le t \le 30$$

$$= 0.5(10 + te^{-\frac{t}{10}}) + 0.6\left(9 + (30 - t)e^{-\frac{30-t}{20}}\right)$$

$$E'(t) = 0.5\left(e^{-\frac{t}{10}} - \frac{1}{10}te^{-\frac{t}{10}}\right) + 0.6\left(-e^{-\frac{30-t}{5}} + \frac{1}{20}(30 - t)e^{-\frac{30-t}{20}}\right)$$

$$= 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{30-t}{5}}(-20 + 30 - t)$$

$$= (0.05e^{-\frac{t}{10}} + 0.03e^{-\frac{30-t}{5}})(10 - t)$$

$E'(t) = 0$ when $10 - t = 0$

$t = 10$ (The first factor is always a positive number.)

$E(0) = 5 + 5.4 + 18e^{-\frac{3}{2}} = 14.42$

$E(10) = 16.65$

$E(30) = 11.15$

For maximum study effectiveness, 10 h of study should be assigned to the first exam and 20 h of study for the second exam.

13. The solution starts in a similar way to that of **12.** The effectiveness function is

$$E(t) = 0.5(10 + te^{-\frac{t}{10}}) + 0.6\left(9 + (25 - t)e^{-\frac{25-t}{20}}\right).$$

The derivative simplifies to

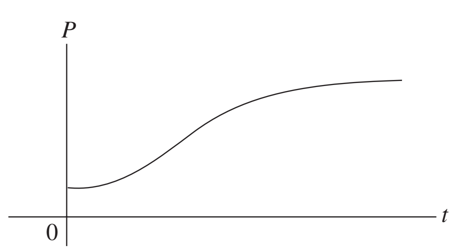
$$E'(t) = 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{25-t}{20}}(5 - t).$$

This expression is very difficult to solve analytically. By calculation on a graphing calculator, we can determine the maximum effectiveness occurs when $t = 8.16$ hours.

14. $P = \dfrac{aL}{a + (L - a)e^{-kLt}}$

a. We are given $a = 100$, $L = 10\ 000$, $k = 0.0001$.

$$P = \frac{10^6}{100 + 9900e^{-t}} = \frac{10^4}{1 + 99e^{-t}} = 10^4(1 + 99e^{-t})^{-1}$$



b. We need to determine when the derivative of the growth rate $\left(\frac{dP}{dt}\right)$ is zero, i.e., when $\frac{d^2P}{dt^2} = 0$.

$$\frac{dP}{dt} = \frac{-10^4(-99e^{-t})}{(1 + 99e^{-t})^2} = \frac{990\ 000e^{-t}}{(1 + 99e^{-t})^2}$$

$$\frac{d^2P}{dt^2} = \frac{-990\ 000e^{-t}(1 + 99e^{-t})^2 - 990\ 000e^{-t}(2)(1 + 99e^{-t})(-99e^{-t})}{(1 + 99e^{-t})^4}$$

$$= \frac{-990\ 000e^{-t}(1 + 99e^{-t}) + 198(990\ 000)e^{-2t}}{(1 + 99e^{-t})^3}$$

$$\frac{d^2P}{dt^2} = 0 \text{ when } 990\ 000e^{-t}(-1 - 99e^{-t} + 198e^{-t}) = 0$$

$$99e^{-t} = 1$$

$$e^t = 99$$

$$t = \ln 99$$

$$\doteq 4.6$$

After 4.6 days, the rate of change of the growth rate is zero. At this time the population numbers 5012.

c. When $t = 3$, $\frac{dP}{dt} = \frac{990\ 000e^{-3}}{(1 + 99e^{-3})^2} \doteq 1402$ cells/day.

When $t = 8$, $\frac{dP}{dt} = \frac{990\ 000e^{-8}}{(1 + 99e^{-8})^2} \doteq 311$ cells/day.

The rate of growth is slowing down as the colony is getting closer to its limiting value.

Exercise 8.5

3. **a.** $y = f(x) = x^x$

$\ln y = x \ln x$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

$$f'(e) = e^e(\ln e + 1) = 2e^e$$

b. $s = e^t + t^e$

$$\frac{ds}{dt} = e^t + et^{e-1}$$

When $t = 2$, $\frac{ds}{dt} = e^2 + e \bullet 2^{e-1}$

c. $y = \dfrac{(x-3)^2\sqrt[3]{x+1}}{(x-4)^5}$

Let $y = f(x)$.

$$\ln y = 2\ln(x-3) + \frac{1}{3}\ln(x+1) - 5\ln(x-4)$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \frac{2}{x-3} + \frac{1}{3(x+1)} - \frac{5}{x-4}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x-3} + \frac{1}{3(x+1)} - \frac{5}{x-4}\right)$$

$$f'(7) = f(7)\left(\frac{2}{4} + \frac{1}{24} - \frac{5}{3}\right)$$

$$= \frac{32}{243}\left(-\frac{27}{24}\right) = -\frac{4}{27}$$

4. $y = x(x^2)$

The point of contact is (2, 16). The slope of the tangent line at any point on the curve is given by $\frac{dy}{dx}$. We take the natural logarithm of both sides and differentiate implicitly with respect to x.

$$y = x^{(x^2)}$$

$$\ln y = x^2 \ln x$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = 2x\ln x + x$$

At the point (2, 16), $\frac{dy}{dx} = 16(4\ln 2 + 2)$.

The equation of the tangent line at (2, 16) is
$y - 16 = 32(2\ln 2 + 1)(x - 2)$.

5. $y = \dfrac{1}{(x+1)(x+2)(x+3)}$

We take the natural logarithm of both sides and differentiate implicitly with respect to x to find $\frac{dy}{dx}$, the slope of the tangent line.

$$\text{In } y = \ln(1) - \ln(x+1) - \ln(x+2) - \ln(x+3)$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = -\frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3}$$

The point of contact is $\left(0, \frac{1}{6}\right)$.

At $\left(0, \frac{1}{6}\right)$, $\frac{dy}{dx} = \frac{1}{6}\left(-1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}\left(-\frac{11}{6}\right) = -\frac{11}{36}$.

6. $y = x^{\frac{1}{x}}, x > 0$

We take the natural logarithm of both sides and differentiate implicitly with respect to x to find $\frac{dy}{dx}$, the slope of the tangent.

$$y = x^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2}$$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2}$$

We want the values of x so that $\frac{dy}{dx} = 0$.

$$\frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2} = 0$$

Since $x^{\frac{1}{x}} \neq 0$ and $x^2 > 0$, we have $1 - \ln x = 0$

$$\ln x = 1$$

$$x = e.$$

The slope of the tangent is 0 at $(e, e^{\frac{1}{e}})$.

7. We want to determine the points on the given curve at which the tangent lines have slope 6. The slope of the tangent at any point on the curve is given by $\frac{dy}{dx} = 2x + \frac{4}{x}$.

To find the required points, we solve:

$$2x + \frac{4}{x} = 6$$

$$x^2 - 3x + 2 = 0$$

$$(x-1)(x-2) = 0$$

$$x = 1 \text{ or } x = 2.$$

The tangents to the given curve at (1, 1) and (2, 4 + 4 ln 2) have slope 6.

8. We first must find the equation of the tangent at A(4, 16). We need $\frac{dy}{dx}$ for $y = x^{\sqrt{x}}$.

$$\text{In } y = \sqrt{x}\ln x$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}\ln x + \sqrt{x} \bullet \frac{1}{x}$$

$$= \frac{\ln x + 2}{2\sqrt{x}}$$

At (4, 16), $\frac{dy}{dx} = 16\frac{(\ln 4 + 2)}{4} = 4\ln 4 + 8$.

The equation of the tangent is
$y - 16 = (4\ln 4 + 8)(x - 4)$.
The y-intercept is $-16(\ln 4 + 1)$.

The x-intercept is $\frac{-4}{\ln 4 + 2} + 4 = \frac{4 \ln 4 + 4}{\ln 4 + 2}$.

The area of ΔOBC is $\frac{1}{2}\left(\frac{4 \ln 4 + 4}{\ln 4 + 2}\right)(16)(\ln 4 + 1)$,

which equals $\frac{32(\ln 4 + 1)^2}{\ln 4 + 2}$.

9. $s(t) = t^{\frac{1}{t}}$, $t > 0$

a. $\ln(s(t)) = \frac{1}{t} \ln t$

Differentiate with respect to t:

$$\frac{1}{s(t)} \bullet s'(t) = \frac{\frac{1}{t} \bullet t - \ln t}{t^2}$$

$$= \frac{1 - \ln t}{t^2}.$$

Thus, $v(t) = s(t) \bullet \left(\frac{1 - \ln t}{t^2}\right) = t^{\frac{1}{t}}\left(\frac{1 - \ln t}{t^2}\right)$.

Now, $a(t) = v'(t) = s'(t)\left(\frac{1 - \ln t}{t^2}\right) + s(t)\left(\frac{-\frac{1}{t} \bullet t^2 - (1 - \ln t)(2t)}{t^4}\right)$.

Substituting for $s(t)$ and $s'(t) = v(t)$ gives

$$a(t) = t^{\frac{1}{t}}\left(\frac{1 - \ln t}{t^2}\right)^2 + t^{\frac{1}{t}}\left(\frac{2t \ln t - 3t}{t^4}\right)$$

$$= \frac{t^{\frac{1}{t}}}{t^4}[1 - 2 \ln t + (\ln t)^2 + 2t \ln t - 3t]$$

b. Since $t^{\frac{1}{t}}$ and t^2 are always positive, the velocity is zero when $1 - \ln t = 0$ or when $t = e$.

$$a(e) = \frac{e^{\frac{1}{e}}}{e^4}[1 - 2 + 1 + 2e - 3e]$$

$$= -\frac{e^{\frac{1}{e}}}{e^3}$$

$$= -e^{\frac{1}{e} - 3}$$

d. $$y = \frac{(x + 2)(x - 4)^5}{(2x^3 - 1)^2}$$

$$\ln y = \ln(x + 2) + 5 \ln(x - 4) - 2 \ln(2x^3 - 1)$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \frac{1}{x + 2} + \frac{5}{x - 4} - \frac{12x^2}{2x^3 - 1}$$

$$\frac{dy}{dx} = \frac{(x + 2)(x - 4)^5}{(2x^3 - 1)^2}\left[\frac{1}{x + 2} + \frac{5}{x - 4} - \frac{12x^2}{2x^3 - 1}\right]$$

f. $y = \left(\sqrt{x^2 + 3}\right)^{e^x}$

$$\ln y = e^x \ln(x^2 + 3)^{\frac{1}{2}}$$

$$= \frac{e^x}{2} \ln(x^2 + 3)$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \frac{e^x}{2} \ln(x^2 + 3) + \frac{e^x}{2} \bullet \frac{2x}{x^2 + 3}$$

$$\frac{dy}{dx} = \left(\sqrt{x^2 + 3}\right)^{e^x}\left[\frac{e^x}{2} \ln(x^2 + 3) + \frac{e^x}{2} \bullet \frac{2x}{x^2 + 3}\right]$$

g. $y = \left(\frac{30}{x}\right)^{2x}$

$$\ln y = 2x[\ln 30 - \ln x]$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = 2[\ln 30 - \ln x] + 2x\left(\frac{-1}{x}\right)$$

$$= \left(\frac{30}{x}\right)^{2x}[2 \ln 30 - 2 \ln x - 2]$$

h. $e^{xy} = \ln(x + y)$

$$e^{xy}\left[x \frac{dy}{dx} + y\right] = \frac{1}{x + y}\left(1 + \frac{dy}{dx}\right)$$

$$\frac{dy}{dx}\left[xe^{xy} - \frac{1}{x + y}\right] = \frac{1}{x + y} + ye^{xy}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x + y} + ye^{xy}}{xe^{xy} - \frac{1}{x + y}} = \frac{1 + (x + y)\, ye^{xy}}{x\,(x + y)e^{xy} - 1}$$

Review Exercise

2. **b.** $y = \frac{x \ln x}{e^x}$

$$\ln y = \ln x + \ln(\ln x) - \ln e^x$$

$$= \ln x + \ln(\ln x) - e^x$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \frac{1}{x} + \frac{1}{\ln x}\left(\frac{1}{x}\right) - 1$$

$$= \frac{x \ln x}{e^x}\left[\frac{1}{x} + \frac{1}{x \ln x} - 1\right]$$

3. b. $f(x) = [\ln(3x^2 - 6x)]^4$

$$f'(x) = 4[\ln(3x^2 - 6x)]^3 \bullet \frac{6x-6}{3x^2-6}$$

Let $f'(x) = 0$, therefore, $\ln(3x^2 - 6x) = 0$

$3x^2 - 6x = 1$

$3x^2 - 6x - 1 = 0$

$$x = \frac{6 \pm \sqrt{48}}{6}$$

$$= \frac{6 \pm 4\sqrt{3}}{6}$$

$$= \frac{3 + 2\sqrt{3}}{3}$$

or

$$\frac{6x-6}{3x^2-6} = 0$$

$6x - 6 = 0$

$x = 1.$

But $3x^2 - 6x > 0$ or $3x(x-2) > 0$.

Therefore, $x > 2$ or $x < 0$.

Only solution is $x = \frac{3 + 2\sqrt{3}}{3}$ and $\frac{3 + 2\sqrt{3}}{3}$.

6. $y = \frac{\ln x^2}{2x}$

$$= \frac{2 \ln x}{x}$$

$$\frac{dy}{dx} = \frac{2\left(\frac{1}{x}\right)x - 2\ln x}{x^2}$$

$$= \frac{2 - 2\ln x}{x^2}$$

At $x = 4$, $\frac{dy}{dx} = \frac{2 - 2\ln 4}{16}$

$$= \frac{1 - \ln 4}{8}.$$

At $x = 4$, $\frac{y = 2\ln 4}{4}$

$$= \frac{\ln 4}{2}.$$

The equation of the tangent is

$$y - \frac{\ln 4}{2} = \frac{1 - \ln 4}{8}(x - 4).$$

$8y - 4\ln 4 = (1 - \ln 4)x - 4 + 4\ln 4$

$(1 - \ln 4)x - 8y + 8\ln 4 - 4 = 0$

7. $y = \frac{e^{2x} - 1}{e^{2x} + 1}$

$$\frac{dy}{dx} = \frac{2e^{2x}(e^{2x}+1) - (e^{2x}-1)(2e^{2x})}{(e^{2x}+1)^2}$$

$$= \frac{2e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x}+1)^2}$$

$$= \frac{4e^{2x}}{(e^{2x}+1)^2}$$

Now, $1 - y^2 = \frac{1 - e^{4x} - 2e^{2x} + 1}{(e^{2x}+1)^2}$

$$= \frac{e^{4x} + 2e^{2x} + 1 - e^{4x} + 2e^{2x} - 1}{(e^{2x}+1)^2}$$

$$= \frac{4e^{2x}}{(3^{2x}+1)^2} = \frac{dy}{dx}$$

8. $y = e^{kx}$

a. $y' - 7y = 0$

$ke^{kx} - 7e^{kx} = 0$

$e^{kx}(k - 7) = 0$

$k = 7$ since $e^{kx} \neq 0$

b. $y'' - 16y = 0$

$k^2e^{kx} - 16e^{kx} = 0$

$e^{kx}(k^2 - 16) = 0$

$k = \pm\ 4$, since $e^{kx} \neq 0$

c. $y'' - y'' - 12y' = 0$

$k^3e^{kx} - k^2e^{kx} - 12ke^{kx} = 0$

$ke^{kx}(k^2 - k - 12) = 0$

$ke^{kx}(k+3)(k-4) = 0$

$k = -3$ or $k = 0$ or $k = 4$, since $e^{kx} \neq 0$

9. The slope of the required tangent line is 3.
The slope at any point on the curve is given by

$$\frac{dy}{dx} = 1 + e^{-x}.$$

To find the point(s) on the curve where the tangent has slope 3, we solve:

$1 + e^{-x} = 3$

$e^{-x} = 2$

$-x = \ln 2$

$x = -\ln 2.$

The point of contact of the tangent is
$(-\ln 2, -\ln 2 - 2)$.
The equation of the tangent line is
$y + \ln 2 + 2 = 3(x + \ln 2)$ or $3x - y + 2\ln 2 - 2 = 0$.

10. The slope of the tangent line to the given curve at any point is

$$\frac{dy}{dx} = \frac{e^x(1 + \ln x) - e^x\left(\frac{1}{x}\right)}{(1 + \ln x)^2}$$

At the point $(1, e)$, the slope of the tangent is $\frac{e-e}{1} = 0$.

Since the tangent line is parallel to the x-axis, the normal line is perpendicular to the x-axis. The line through $(1, e)$ perpendicular to the x-axis has equation $x = 1$.

11. a. $\frac{dN}{dt} = 2000\left[e^{-\frac{t}{20}} - \frac{1}{20}te^{-\frac{t}{20}}\right]$

$= 2000e^{-\frac{t}{20}}\left[1 - \frac{t}{20}\right]$

Since $e^{-\frac{t}{20}} > 0$ for all t, $\frac{dN}{dt} = 0$,

when $1 - \frac{t}{20} = 0$

$t = 20.$

The growth rate of bacteria is zero bacteria per day on day 20.

b. When $t = 10$, $N = 2000[30 + 10e^{-\frac{1}{2}}]$

$\doteq 72\ 131$

$m = \sqrt[3]{72\ 131 + 1000}$

$\doteq 41.81.$

On day 10, there will be 42 newly infected mice.

12. $g(t) = \frac{\ln(t^3)}{2t}$

$= \frac{3 \ln t}{2}, t > 1$

$g'(t) = \frac{\frac{3}{t} \bullet 2t - (3 \ln t)(2)}{4t^2}$

$= \frac{6 - 6 \ln t}{4t^2}$

Since $t > 1$, $g'(t) - 0$ when $6 - 6 \ln t = 0$

$\ln t = 1$

$t = e.$

Now, $\lim_{t \to 1^+} g(t) = \lim_{t \to 1^+} \left(\frac{3 \ln (t)}{2t}\right) = 0$

$g(e) = \frac{3}{23}$

$\doteq 0.552$

For $t > e$, $\ln t > 1$ and $g'(t) < 0$

Thus, the maximum measure of effectiveness of this medicine is 0.552 and occurs at $t = 2.718$ h after the medicine was given.

13. $m(t) = t \ln (t) + 1$ for $0 < t \leq 4$

$m'(t) = \ln (t) + 1$

$m'(t) = 0$ when $\ln (t) + 1 = 0$

$t = e^{-1}$

For $0 < t < e^{-1}$, $m'(t) < 0$.

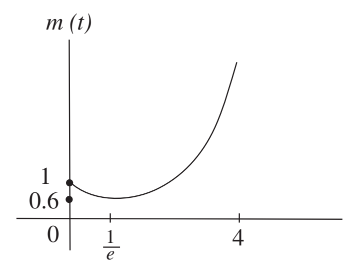
Thus, $m(t)$ is decreasing over this interval.

$\lim_{t \to 0^+} (t \ln t + 1) = 1$ (by investigating the graph of $m(t)$)

$m(e^{-1}) \doteq .632$

$m(4) \doteq 6.545$

During the first four years, a child's ability to memorize is lowest at 0.368 years of age and highest at four years.



14. a. $c_1(t) = te^{-t}$; $c_1(0) = 0$

$c_1'(t) = e^{-t} - te^{-t}$

$= e^{-t}(1 - t)$

Since $e^{-t} > 0$ for all t, $c_1'(t) = 0$ when $t = 1$.

Since $c_1'(t) > 0$ for $0 \leq t < 1$, and $c_1'(t) < 0$ for all $t > 1$, $c_1(t)$ has a maximum value of $\frac{1}{e} \doteq 0.368$ at $t = 1$ h.

$c_2(t) = t^2e^{-t}$; $c_2(0) = 0$

$c_2'(t) = 2te^{-t} - t^2e^{-t}$

$= te^{-t}(2 - t)$

$c_2'(t) = 0$ when $t = 0$ or $t = 2$.

Since $c_2'(t) > 0$ for $0 < t < 2$ and $c_2'(t) < 0$ for all $t > 2$, $c_2(t)$ has a maximum value of $\frac{4}{e^2} \doteq 0.541$ at $t = 2$ h. The larger concentration occurs for drug c_2.

b. $c_1(0.5) = 0.303$
$c_2(0.5) = 0.152$
In the first half-hour, the concentration of c_1 increases from 0 to 0.303, and that of c_2 increases from 0 to 0.152. Thus, c_1 has the larger concentration over this interval.

15. $T(x) = 10\left(1 + \frac{1}{x}\right)(0.9)^{-x}$

a. $T'(x) = 10\left(-\frac{1}{x^2}\right)(0.9)^{-x} + 10(1 + \frac{1}{x})(0.9)^{-1}(-1)(\ln(0.9))$

$$= 10(0.9)^{-x}\left[-\frac{1}{x^2} - \ln(0.9)\left(1 + \frac{1}{x}\right)\right]$$

b. Since $(0.9)^{-x} > 0$ for all x, $T'(x) = 0$ when

$$-\frac{1}{x^2} - \ln(0.9) - \frac{\ln(0.9)}{x} = 0.$$

To find an approximate solution, we use $l(0.9) \doteq -0.1$. The quadratic equation becomes

$$-\frac{1}{x^2} + 0.1 + \frac{0.1}{x} = 0$$

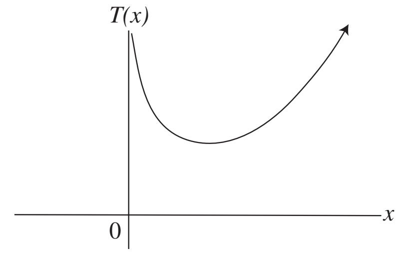
$0.1x^2 + 0.1x - 1 = 0, x \neq 0$

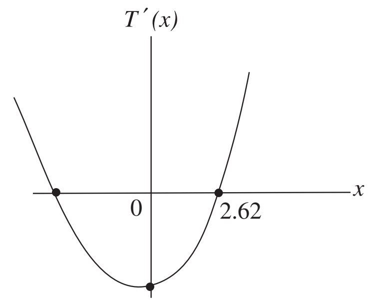
$x^2 + x - 10 = 0$

$$x = \frac{-1 \pm \sqrt{1 + 40}}{2}$$

$= 2.7$, since $x \geq 0$.

Note: Using $\ln(0.9) \doteq -0.105$ yields $x \doteq 2.62$.
Since $T'(x) < 0$ for $0 < x < 2.62$, and $T'(x) > 0$ for $x > 2.62$, $T(x)$ has a minimum value at $x = 2.62$.





16. $v(x) = Kx^2 \ln\left(\frac{1}{x}\right)$

a. $v(x) = 2x^2 \ln\left(\frac{1}{x}\right) = -2x^2 \ln x$

$v\left(\frac{1}{2}\right) = 2\left(\frac{1}{4}\right)(\ln 2)$

$= \frac{\ln 2}{2}$

$= 0.347$

b. $v'(x) = -4x \ln x - 2x^2\left(\frac{1}{x}\right)$

$= -4x \ln x - 2x$

$v'\left(\frac{1}{2}\right) = 2 \ln 2 - 1$

$\doteq 1.386$

17. $C(t) = K(e^{-2t} - e^{-5t})$

a. $\lim_{t\to\infty} C(t) = \lim_{t\to\infty} K\left(\frac{1}{e^{2t}} - \frac{1}{e^{5t}}\right)$

$= K(0 - 0)$

$= 0$

b. $C'(t) = K(-2e^{-2t} + 5e^{-5t})$

$$C'(t) = 0 \text{ when } -\frac{2}{e^{2t}} + \frac{5}{e^{5t}} = 0$$

$$\frac{5}{e^{5t}} = \frac{2}{e^{2t}}$$

$$\frac{5}{2} = e^{5t-2t} = e^{3t}$$

$$3t = \ln\left(\frac{5}{2}\right)$$

$$t = \frac{\ln\left(\frac{5}{2}\right)}{3} \doteq 0.305$$

The rate is zero at $t = 0.305$ days or 7.32 h.

Chapter 8 Test

1. $y = e - 2x^2$

a. $\frac{dy}{dx} = -4xe^{-2x^2}$

b. $y = \ln(x^2 - 6)$

$$\frac{dy}{dx} = \frac{1}{x^2 - 6} \bullet 2x = \frac{2x}{x^2 - 6}$$

c. $y = 3^{x^2 + 3x}$

$$\frac{dy}{dx} = 3x^2 + 3x \bullet \ln 3 \bullet (2x + 3)$$

d. $y = \frac{e^{3x} + e^{-3x}}{2}$

$$\frac{dy}{dx} = \frac{1}{2}[3e^{3x} - 3e^{-3x}]$$

$$= \frac{3}{2}[e^{3x} - e^{-3x}]$$

e. $y = (4x^3 - x)\log_{10}(2x - 1)$

$$\frac{dy}{dx} = (12x^2 - 1)\log_{10}(2x - 1) + (4x^3 - x) \bullet \frac{1}{(2x - 1)\ln 10} \bullet 2$$

f. $y = \frac{\ln(x + 4)}{x^3}$

$$\frac{dy}{dx} = \frac{\frac{1}{x + 4} \bullet x^3 - \ln(x + 4) \bullet 3x^2}{x^6}$$

$$= \frac{\frac{x}{x + 4} - \ln(x + 4)}{x^4}$$

2. $f(t) = \ln(3t^2 + t)$

$$f'(t) = \frac{1}{3t^2 + t} \bullet (6t + 1)$$

Thus, $f'(2) = \frac{13}{14}$.

3. $y = x^{\ln x}$, $x > 0$

To find $\frac{dy}{dx}$, we take the natural logarithm of both sides and differentiate implicitly with respect to x.

$$y = x^{\ln x}$$

$$\ln y = \ln x \ln x = (\ln x)^2$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = 2 \ln x \bullet \frac{1}{x}$$

The point of contact is (e, e).

At this point, $\frac{1}{e} \bullet \frac{dy}{dx} = 2 \bullet \frac{1}{e}$

$$\frac{dy}{dx} = 2.$$

The slope of the tangent at (e, e) is 2.

4. $x^2y + x\ln x = 3y$, $x > 0$

We differentiate implicitly with respect to x.

$$2xy + x^2\frac{dy}{dx} + \ln x + x \bullet \frac{1}{x} = 3\frac{dy}{dx}$$

$$2xy + \ln x + 1 = \frac{dy}{dx}(3 - x^2)$$

$$\frac{dy}{dx} = \frac{2xy + \ln x + 1}{3 - x^2}$$

Alternate Solution

y can be expressed explicitly as a function of x.

$$y = \frac{x \ln x}{3 - x^2}$$

$$\frac{dy}{dx} = \frac{(\ln x + 1)(3 - x^2) - x \ln x\,(-2x)}{(3 - x^2)^2}$$

$$= \frac{x^2 \ln x + 3 \ln x - x^2 + 3}{(3 - x^2)^2}$$

5. Since $e^{xy} = x$, $xy = \ln x$.

$$y = \frac{\ln x}{x}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x} \bullet x - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

At $x = 1$, $\frac{dy}{dx} = \frac{1 - \ln 1}{1} = 1$.

Alternate Solution

$e^{xy} = x$

We differentiate implicitly with respect to x

$$e^{xy}\left(y + x\frac{dy}{dx}\right) = 1$$

$$x\frac{dy}{dx} = \frac{1}{e^{xy}} - y$$

When $x = 1$, $y = 0$.

Thus, $\frac{dy}{dx} = \frac{1}{e^0} - 0 = 1$

6. $y'' + 3y' + 2y = 0$

$y = e^{Ax}$, $y' = Ae^{Ax}$, $y'' = A^2e^{Ax}$

The differential equation is

$A^2e^{Ax} + 3Ae^{Ax} + 2e^{Ax} = 0$

$e^{Ax}(A^2 + 3A + 2) = 0$

$(A + 1)(A + 2) = 0$, $e^{Ax} \neq 0$

$A = -1$ or $A = -2$.

7. The slope of the tangent line at any point on the curve is given by $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 3^x \ln 3 + \ln x + 1$$

At $A(1, 3)$, the slope of the tangent is $3 \ln 3 + 1$.

The slope of the normal line is $-\frac{1}{3 \ln 3 + 1}$.

The equation of the normal line is

$$y - 3 = -\frac{1}{3 \ln 3 + 1}(x - 1).$$

8. $v(t) = 10e^{-kt}$

a. $a(t) = v'(t) = -10ke^{-kt}$

$= -k(10e^{-kt})$

$= -kv(t)$

Thus, the acceleration is a constant multiple of the velocity. As the velocity of the particle decreases, the acceleration increases by a factor of k.

b. At time $t = 0$, $v = 10$ cm/s.

c. When $v = 5$, we have $10e^{-kt} = 5$

$$e^{-kt} = \frac{1}{2}$$

$$-kt = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$t = \frac{\ln 2}{k}.$$

After $\frac{\ln 2}{k}$ s have elapsed, the velocity of the particle is 5 cm/s. The acceleration of the particle is $-5k$ at this time.

9. We have Profit = Revenue – Cost:

$P(p) = 4000[e^{0.01(p-100)} + 1] - 50p$, $100 \leq p \leq 250$

We apply the Algorithm for Extreme Values:

$P'(p) = 4000[e^{0.01(p-100)}(0.01)] - 50.$

For critical values, we solve $P'(p) = 0$

$$40e^{0.01(p-100)} - 50 = 0$$

$$e^{0.01(p-100)} = \frac{5}{4}$$

$$0.01(p - 100) = \ln(1.25)$$

$$p - 100 = 100 \ln(1.25)$$

$$p = 100 \ln(1.25) + 100$$

$$\doteq 122.3.$$

Since the number of jackets produced must be an integer, we evaluate P for $p = 100$, 122, 123, and 250.

$P(100) = 3000$

$P(122) = 2884.81$

$P(123) = 2884.40$

$P(250) = 9426.76$

The maximum profit of \$9426.76 occurs when 250 jackets are produced and sold. The price per jacket is given by Revenue ÷ number of jackets. Thus, selling price per jacket is

$$\frac{R(250)}{250} = \frac{21\,926.76}{250}$$

$$= \$87.71.$$